Optimal Selection of Intermediate Storage Tank Capacity in a Periodic Batch/Semicontinuous Process

Batch/semicontinuous chemical plants are usually designed either by assuming infinite intermediate storage or by assuming that the units themselves act as storage vessels, while the storage vessels are sized by rules of thumb or experience. In this paper, the case of an intermediate storage vessel which links one upstream batch/semicontinuous unit to one downstream batch/semicontinuous unit is analyzed. The units are assumed to operate with fixed cycle times and capacities. Expressions for determining the minimum storage tank capacity necessary to decouple the two units are derived from a mathematical model of the periodic process. Effects of the relative starting times of the two units on the required storage capacity are determined, thus suggesting the optimum process timings to minimize the same. Application of the results is illustrated by an example.

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SCOPE

Storage tanks often represent a sizable portion of a chemical plant and serve a variety of purposes (Ross, 1973; Henley and Hoshino, 1977). One primary use of intermediate storage tanks is to act as surge vessels decoupling the upstream and downstream units to maintain continuity of operation. They are particularly useful in the case of multiproduct batch/semicontinuous processes in which they are also used to free up any expensive or time-consuming unit or to affect a change in product sequencing. Generally, the storage tanks are sized by experience or by a variety of rules of thumb, which obviously may lead to the disadvantages associated with either too little or too much storage. Batch/semicontinuous processes, whose examples abound in the food, pharmaceutical, polymer and fine chemicals processing industries, have been usually designed either by assuming infinite intermediate storage or by assuming that the units themselves act as their own storage vessels. Incorporation of some rational ways of sizing intermediate storage in these design procedures (e.g., Knopf et al., 1980) can definitely lead to more efficient plant designs with lesser costs. Apart from a few attempts, a systematic approach to this important problem of the design of storage tanks is surprisingly absent in the literature.

Ross (1973) illustrated the use of Monte-Carlo simulation as a tank-sizing method. Oi et al. (1977) considered the optimal scheduling and sizing of intermediate storage in a waste treatment system for periodically fluctuating flows from multiple sources. Takamatsu et al. (1979) treated a case of storage intermediate to one continuous unit and N parallel batch units and analytically derived the relationship between minimum storage tank capacities and optimal process timing of each batch unit. Recently, Takamatsu et al. (1982) investigated the problem of optimal design and operation of a single product batch process with multiple batch stages and intermediate storage tanks. As part of that analysis, they obtained results on the maximum interstage storage for several cases of the serial batch-batch configuration.

In this paper, we deal with the general case of a storage vessel intermedidate to two batch-semicontinuous units which operate periodically with fixed cycle times and batch sizes. First of all, we derive a mathematical model of the periodic process. The analysis is divided into two cases. We first consider the case in which the inlet flow rate to the storage vessel is greater than or equal to the outlet flow rate from the storage vessel. Then, we consider the case in which opposite is true. For both cases, expressions for the maximum holdup in the storage vessel are derived as a function of the delay time of the downstream unit with respect to the upstream unit and other characteristic sizes, flow rates and cycle times of the equipment. The differences in dealing with various arrangements of the units are briefly mentioned. Effects of the choice of delay time on the required minimum storage capacity are also outlined. Finally, application of the results is illustrated by an example.

CONCLUSIONS AND SIGNIFICANCE

The case of a storage vessel intermediate to two batch/semicontinuous units which operate periodically with fixed batch sizes and cycle times has been analyzed. Expressions for calculating the minimum storage capacity necessary for decoupling the upstream and downstream units to maintain the continuity of the operation are presented.

In addition to the characteristic batch sizes, flow rates and cycle times of the system, the difference between the initial starting moments of the two units also affects the required tank size. Tank size has been shown to be a piecewise linear and periodic function of the above-mentioned time difference (delay time). At its maximum and minimum values, this function has either a zero slope or a corner point, while at all its intermediate values it either increases or decreases linearly with respect to the delay time. Optimum delay time values which would minimize the required storage capacity have been suggested. There is a range of values of delay time over which the operation can be started without any initial holdup in the intermediate storage vessel and such that the continuity of operation can be maintained, while for all other values of the delay time operation has to be started with an initial inventory predicted by the equations. Apart from being able to calculate the tank size, one can also determine the maximum holdup possible in an intermediate storage vessel given the value of the delay time, and initial holdup in the storage vessel or alternately choose a suitable delay time to maintain the continuity of the operation given the initial holdup and the size of storage vessel. As shown, the expressions are applicable to any one-to-one arrangement of periodically operating batch/semicontinuous/continuous units regardless of their order or type (e.g., batch-batch, batch-continuous, continuous-batch etc.), batch sizes, flow rates and process timings.

This work is a step towards the achievement of more rational ways of sizing intermediate storage vessels. It also gives a basis for evaluating whether existing storage facilities have either too little or too much storage. Allowance for uncertainties can be added above the prescribed minimum capacity depending on the situation. Moreover, the results can be very easily incorporated into existing procedures for designing batch/semicontinuous processes which do not take into account the existence of intermediate storage vessels either in the operation or in the system cost. The undetermined variables such as batch sizes, flow rates and cycle times can be suitably selected in an overall optimization scheme of the whole plant. In addition, the existing procedures for designing batch/semicontinuous processes do not allow multiple batch sizes for a given product in a designed plant. Introduction of intermediate storage vessels between the processing stages allows the freedom to choose different batch sizes for different subprocesses separated by intermediate storage vessels. The results of this paper facilitate the sizing of such intermediate storage vessels; hence such overall optimization with above-mentioned flexibility will lead to plant designs with better percent utilization of the equipment, increased throughput and reduced cost. In essence, the results are useful both in the operation of existing plants and in the design of new plants.

MATHEMATICAL MODEL OF THE PROCESS

Assumptions

In the process configuration shown in Figure 1, V_1 and V_2 are the batch sizes of two batch/semicontinuous units; V^* is the size of the intermediate storage vessel; and ω_1 and ω_2 are the cycletimes of the two units. The underlying assumptions used in formulating the mathematical model of the process are:

(1) Semicontinuous units process material in batches of fixed sizes at constant rate. After processing a fixed amount of material, they remain idle for a fixed interval of time and then resume operation in a periodic manner.

(2) Batch units operate with fixed batch sizes and fixed cycle times. One cycle of a batch unit consists of operations such as filling, processing, emptying, and preparation and/or waiting.

(3) The productivities of both units are equal; i.e., $V_1/\omega_1 = V_2/\omega_2$.

(4) There exist least integers β_m and β'_m such that $\beta_m\omega_1 = \beta'_m\omega_2$. Essentially, ω_1 and ω_2 can as an engineering approximation be rounded off to a finite number of significant figures.

(5) The flow rates into and out of the intermediate storage vessel, U_1 and U_2 respectively, are constant.

(6) The required size of the storage tank is equal to the maximum holdup in the tank.

Formulation

Define a capacity ratio, m, given by,

$$m = V_1/V_2 = \omega_1/\omega_2 \tag{1}$$

The cycle time of a batch unit, ω, is given by the expression, $\omega = T_f + T_B + T_e + T_p$. The cycle time of a semicontinuous unit, ω, is similarly given by, $\omega = T_S + T_i$. For batch units, we define emptying and filling fractions, x_1 and x respectively, as follows,

$$x_1 = (T_e)_1/\omega_1 \text{ and } x = (T_f)_2/\omega_2 \tag{2a,b}$$

For semicontinuous units, we define similar quantities which are equivalent to processing fractions,

$$x_1 = (T_S)_1/\omega_1 \text{ and } x = (T_S)_2/\omega_2 \tag{2c,d}$$

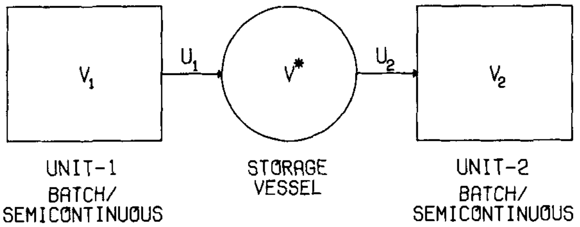

Figure 1. Schematic diagram of the process.

Now, $V_1 = U_1(T_e)_1$ and $V_2 = U_2(T_f)_2$; therefore, $V_1 = U_1x_1\omega_1$ and $V_2 = U_2x\omega_2$. Defining the ratio of input and outlet flowrates, r, to be,

$$r = U_1/U_2 \tag{3a}$$

we obtain, from the preceding relations,

$$x_1 = x/r \tag{3b}$$

If we select as the origin of time, the time when the storage tank commences to be filled from Unit 1, the input and output functions to and from the storage tank, $F_1(t)$ and $F_2(t)$ respectively, are given as follows,

$$F_1(t) = \begin{Bmatrix} U_1 i\omega_1 \le t \le i\omega_1 + x_1\omega_1 \\ 0 i\omega_1 + x_1\omega_1 < t < (i+1)\omega_1 \end{Bmatrix} \text{ where } i \text{ is an integer}$$

$$F_2(t) = \begin{Bmatrix} U_2 j\omega_2 \le t \le j\omega_2 + x\omega_2 \\ 0 j\omega_2 + x\omega_2 < t < (j+1)\omega_2 \end{Bmatrix} \text{ where } j \text{ is an integer}$$

Next, we define the delaytime, t_0, as the difference between the moment when Unit 1 commences being emptied for the first time and the moment when Unit 2 commences being filled for the first time.

The fractional delay time variable, y, is defined as,

$$y = t_0/\omega_2 \tag{4}$$

Since, $F_2(t)$ has a period of ω_2, the range of y is $0 \le y < 1$. Finally, the holdup in the intermediate storage vessel, $V(t)$, is described by,

$$\frac{dV(t)}{dt} = F_1(t) - F_2(t - t_0) \tag{5}$$

Our aim is to solve this first-order ordinary differential equation with discontinuous righthand side to evaluate the maximum holdup in the storage tank in terms of the fractional delay time variable y and other variables such as batchsizes, capacity ratio, flow rates and cycle times, subject to the condition that $V(t) \ge 0$. Having the solution, we also wish to find out the optimum value of the variable y so that maximum holdup is minimized. We resort to the Fourier Series (Tuma, 1979) as the tool for compactly expressing the discontinuous righthand side of Eq. 5.

The step function $F_0(t)$, shown in Figure 2, which has a period ω_0, can be represented in the form of a Fourier series, as follows,

$$F_0(t) = \frac{V_0}{\omega_0} + \sum_{n=1}^{\infty} \frac{2U_0}{n\pi} \operatorname{Sin} n\pi x_0 \cos 2n\pi \left(\frac{t}{\omega_0} - \frac{x_0}{2}\right)$$

where

$$V_0 = U_0 x_0 \omega_0$$

Representing both $F_1(t)$ and $F_2(t - t_0)$ via the above Fourier series and using Assumption 3, Eqs. 3a and 4, we get,

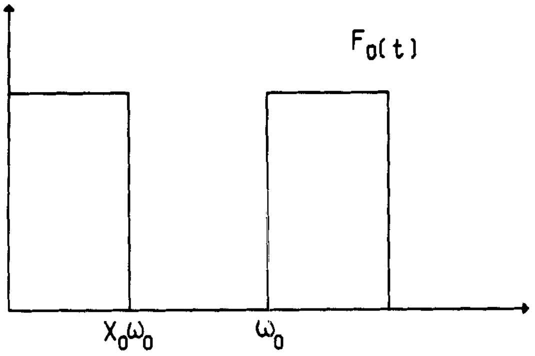

Figure 2. A typical flow rate function $F_0(t)$.

$$F_1(t) - F_2(t - t_0) = \sum_{n=1}^{\infty} \frac{2U_2}{n\pi} \times \left[r \operatorname{Sin} n\pi x_1 \operatorname{Cos} 2n\pi \left(\frac{t}{\omega_1} - \frac{x_1}{2} \right) - \sin n\pi x \cos 2n\pi \left(\frac{t}{\omega_2} - y - \frac{x}{2} \right) \right] \quad (6)$$

Integrating Eq. 5 once, we obtain,

$$V(t) = V(0) + I(t) \quad (7)$$

where, $I(t)$ is defined as,

$$I(t) = \int_0^t F_1(\tau) - F_2(\tau - t_0) d\tau \quad (8)$$

Integrating both sides of Eq. 6

$$I(t) = \sum_{n=1}^{\infty} \frac{U_2\omega_2}{n^2\pi^2} \left[mr \operatorname{Sin}^2 n\pi x_1 + mr \operatorname{Sin} n\pi x_1 \cos 2n\pi \left(\frac{t}{\omega_1} - \frac{x_1}{2} \right) - \operatorname{Sin} n\pi x \operatorname{Sin} 2n\pi \left(\frac{t}{\omega_2} - y - \frac{x}{2} \right) - \operatorname{Sin} n\pi x \operatorname{Sin} 2n\pi \left(y + \frac{x}{2} \right) \right]$$

$$\sum_{n=1}^{\infty} \frac{\operatorname{Cos} 2n\pi z}{n^2\pi^2} = \frac{1}{6} - |z| + z^2 \quad |z| \leq 1 \quad (9)$$

Since, $0 < x_1 \leq 1$, using Eq. 3b, Eq. 9 and the relation $V_2 = U_2 x \omega_2$, we obtain

$$\sum_{n=1}^{\infty} \frac{mrU_2\omega_2}{n^2\pi^2} \operatorname{Sin}^2 n\pi x_1 = \frac{m}{2}(1 - x_1)V_2 \quad (10)$$

Therefore, $I(t)$ can be rewritten as,

$$I(t) = \frac{m}{2}(1 - x_1)V_2 + \sum_{n=1}^{\infty} \frac{U_2\omega_2}{n^2\pi^2} \times \left[mr \operatorname{Sin} n\pi x_1 \operatorname{Cos} 2n\pi \left(\frac{t}{\omega_1} - \frac{x_1}{2} \right) - \operatorname{Sin} n\pi x \operatorname{Cos} 2n\pi \left(\frac{t}{\omega_2} - y - \frac{x}{2} \right) - \operatorname{Sin} n\pi x \operatorname{Sin} 2n\pi \left(y + \frac{x}{2} \right) \right] \quad (11)$$

From Eqs. 11 and 7, we derive the following proposition.

Proposition I. The holdup in the intermediate storage vessel, $V(t)$, is given by,

$$V(t) = V(0) + \frac{m}{2}(1 - x_1)V_2 + \sum_{n=1}^{\infty} \frac{U_2\omega_2}{n^2\pi^2} \left[mr \operatorname{Sin} n\pi x_1 \operatorname{Cos} 2n\pi \left(\frac{t}{\omega_1} - \frac{x_1}{2} \right) - \operatorname{Sin} n\pi x \operatorname{Cos} 2n\pi \left(\frac{t}{\omega_2} - y - \frac{x}{2} \right) - \operatorname{Sin} n\pi x \operatorname{Sin} 2n\pi \left(y + \frac{x}{2} \right) \right]$$

Note that since, Cosine has a period of 2π, it follows from Assumption 4 that, $V(t) = V(t + \beta_m\omega_1) = V(t + \beta'_m\omega_2)$. Therefore, we can state the following proposition about the nature of $V(t)$.

Proposition II. The holdup in the intermediate storage vessel, $V(t)$, is a periodic function with a period of $\beta_m\omega_1$ which is also equal to $\beta'_m\omega_2$.

Proposition II is very useful in evaluating the maximum holdup since, it lets us to concentrate our attention only on the range, $0 \leq t < \beta_m\omega_1$. For this purpose, we define the following quantities,

$$V_{\max} = \max_{0 \leq t < \beta m \omega_1} I(t) \quad \text{and} \quad V_{\min} = \min_{0 \leq t < \beta m \omega_1} I(t) \quad (12a,b)$$

From Eq. 7, it follows that

$$\max_{0 \leq t < \beta m \omega_1} V(t) = V(0) + V_{\max}$$

and

$$\min_{0 \leq t < \beta m \omega_1} V(t) = V(0) + V_{\min} = 0, \text{ since negative}$$

holdup is infeasible. Hence, from Assumption 6,

$$V^* = \max_{0 \leq t < \beta m \omega_1} V(t) = V_{\max} - V_{\min} \quad (13)$$

Hence, from Eqs. 12a and 12b, we get,

$$V_{\max} = \max_{0 \leq \alpha \leq \beta_m - 1} V^+(\alpha) \quad \text{and} \quad V_{\min} = \min_{0 \leq \beta \leq \beta_m - 1} V^-(\beta) \quad (15a,b)$$

Using Eqs. 11, 14a and 14b, we can demonstrate the following proposition. (For proof, see Appendix I.)

Proposition IIIa. The local maxima and the local minima of $I(t)$ for $r \geq 1$, are given by,

$$V^+(\alpha) = m(1 - x_1)V_2 - \sum_{n=1}^{\infty} \frac{U_2\omega_2}{n^2\pi^2} \operatorname{Sin} n\pi x \times \left[\operatorname{Sin} 2n\pi \left(\alpha p + \delta_1 - y - \frac{x}{2} \right) + \operatorname{Sin} 2n\pi \left(y + \frac{x}{2} \right) \right]$$

and

$$V^-(\beta) = -\sum_{n=1}^{\infty} \frac{U_2\omega_2}{n^2\pi^2} \operatorname{Sin} n\pi x \times \left[\operatorname{Sin} 2n\pi \left(\beta p - y - \frac{x}{2} \right) + \operatorname{Sin} 2n\pi \left(y + \frac{x}{2} \right) \right]$$

respectively, where, $p = 1/\beta_m$ and δ_1 is defined by the expression, $mx_1 = kp + \delta_1$, such that $0 \leq \delta_1 < p$ for an integer k.

We also can state (proof in Appendix II) the following important proposition about the nature of $V^*(y)$.

Proposition IV. The size of the storage vessel, $V^*(y)$, is a periodic function of y with period p.

Next, we proceed to develop the expressions for $V_{\max}$ and $V_{\min}$ using Eqs. 15a and 15b. Here, we will present only the development of expressions for $V_{\min}$. The development of the expressions for $V_{\max}$ is very similar, hence only the main differences will be pointed out.

EVALUATION OF V_{max} AND V_{min}

In this section, we will develop the expressions for $V_{\max}$ and $V_{\min}$ for the determination of the size of the storage vessel. We divide this section into two parts. In the first subsection, we consider the case in which $U_1 \geq U_2$ or $r \geq 1$, while in the next subsection we consider the case in which $U_1 < U_2$ or $r < 1$. Before proceeding to these subsections, we state without proof a lemma which is a direct consequence of the discrete nature of the derivative of $V(t)$. The function $V(t)$ for the example to be discussed later is shown in Figure 6

Lemma I. $V(t)$ is a piecewise linear function with multiple optima all of which are corner points.

Case I: $r \geq 1$.

For this case, since $U_1 \geq U_2$, it is easy to see that,

$$\frac{dV(t)}{dt} \geq 0, \quad i\omega_1 < t \leq i\omega_1 + x_1\omega_1$$

and

$$\frac{dV(t)}{dt} \leq 0, \quad i\omega_1 + x_1\omega_1 < t \leq (i+1)\omega_1$$

where i is any integer.

Let t_{max} and t_{min} be the times at which $I(t)$ takes on local maximum and local minimum values respectively. Since, we can restrict consideration to the domain $0 < t \leq \beta_m\omega_1$, it can be shown that,

$$t_{max} = \alpha\omega_1 + x_1\omega_1 \quad 0 \leq \alpha \leq \beta_m - 1 \tag{14a}$$

and

$$t_{min} = \beta\omega_1 \quad 0 \leq \beta \leq \beta_m - 1 \tag{14b}$$

where α and β are integer variables. Let $V^+(\alpha)$ and $V^-(\beta)$ represent the values of these local maxima and local minima in $I(t)$ respectively.

First of all, we express $V^-(\beta)$ of Proposition IIIa in terms of cosine functions as follows,

$$V^-(\beta) = -\sum_{n=1}^{\infty} \frac{U_2\omega_2}{2n^2\pi^2}[\text{Cos } 2n\pi(\beta p - y - x) - \text{Cos } 2n\pi(\beta p - y) + \text{Cos } 2n\pi y - \text{Cos } 2n\pi(x+y)]$$

This is an essential intermediate step because now we can use Eq. 9 to express the sum in a finite form. Next, we modify the arguments of the first two terms by applying the transformation, $x + y = k_1p + y'$ such that $0 \leq y' < p$ for an integer k_1. For V_{max}, we apply the transformation, $y = k_2p + \delta_1 + y'$ such that $0 \leq y' < p$ for an integer k_2. Now, $\beta_m p = 1$ and the fact that cosine has a period 2π imply that $(\beta - k_1)p$ can be equivalently represented by $\beta' p$ such that $0 \leq \beta' \leq \beta_m - 1$. Therefore, we obtain

$$V^-(\beta) = -\sum_{n=1}^{\infty} \frac{U_2\omega_2}{2n^2\pi^2}[\text{Cos } 2n\pi(\beta' p - y') - \text{Cos }(\beta' p - y' + x) + \text{Cos } 2n\pi y - \text{Cos } 2n\pi(x+y)]$$

Subtracting $\beta_m p = 1$ from the first two terms and denoting $(\beta' - \beta_m)p$ again by $-\beta p$, $0 \leq \beta \leq \beta_m - 1$, we obtain,

$$V^-(\beta) = -\sum_{n=1}^{\infty} \frac{U_2\omega_2}{2n^2\pi^2}[\text{Cos } 2n\pi(\beta p + y') - \text{Cos }(\beta p + y' - x) + \text{Cos } 2n\pi y - \text{Cos } 2n\pi(x+y)]$$

If we restrict our attention to the range $y \leq 1 - x$, it can be seen that the absolute values of all the arguments (exclusive of the factor of $2n\pi$) of the Cosine functions are less than or equal to unity. Therefore, using Eq. 9, the above sum reduces to,

$$V^-(\beta) = \frac{U_2\omega_2}{2}[(\beta p + y' - x) - |\beta p + y' - x| - 2x\beta p + 2xk_1p]$$

To obtain an expression for V_{min}, we divide the range of β values, $0 \leq \beta \leq \beta_m - 1$, into two parts: (1) $\beta p + y' - x \geq 0$; (2) $\beta p + y' - x < 0$.

For these ranges, we have

$$V^-(\beta) = U_2\omega_2(k_1px - \beta px), \; \beta p + y' - x \geq 0$$

$$V^-(\beta) = U_2\omega_2[y' - x(1 - k_1p) + \beta p(1-x)], \quad \beta p + y' - x < 0$$

It is clear from these expressions that $V^-(\beta)$ takes on minimum values at $\beta = \beta_m - 1$ and $\beta = 0$ for the above two ranges, respectively. We are required to choose the minimum of these two values which are themselves minima in their respective ranges. One can easily conclude that,

$$V_{min} = U_2\omega_2[px - x(1 - k_1p)] \qquad y' \geq px$$

$$V_{min} = U_2\omega_2[y' - x(1 - k_1p)] \qquad y' \leq px$$

We can derive similar expressions for the case $y \geq 1 - x$, since $\text{Cos } 2n\pi(x + y)$ can be written as $\text{Cos } 2n\pi[y - (1 - x)]$ and so Eq. 9 can again be applied. For $y \geq 1 - x$, we obtain the results,

$$V_{min} = U_2\omega_2[-y' + px + (1-x)(1 - k_1p)] \qquad y' \geq px$$

$$V_{min} = U_2\omega_2(1-x)(1 - k_1p) \qquad y' \leq px$$

These formulae and the corresponding results for V_{max} can be expressed in the more convenient form shown in Proposition V.

Proposition V. The expressions for V_{min} and V_{max} for $r > 1$, are

a) $V_{min} = U_2\omega_2[z_1 - x(1 - k_1p)]; \qquad y \leq 1 - x$

$V_{min} = U_2\omega_2[-y' + z_1 + (1-x)(1 - k_1p)]; \quad y \geq 1 - x$

where, $z_1 = \min[y', px]$ and y' is defined by the expression, $x + y = k_1p + y'$, such that $0 \leq y' < p$ for an integer k_1.

b) $V_{max} = m(1 - x_1)V_2 + U_2\omega_1[-z_2 + px + x(k_2p + \delta_1)]; \qquad y \leq 1 - x$

$V_{max} = m(1 - x_1)V_2 + U_2\omega_1[-y' - z_2 + px + (1-x)(1 - k_2p - \delta_1)]; \quad y \geq 1 - x$

where $z_2 = \min[p - y', px]$ and y' is defined by the expression, $y = k_2p + \delta_1 + y'$ such that $0 \leq y' < p$ for an integer k_2.

Case II: $r < 1$.

The analysis for this case is similar in strategy to the previous case but differs in some of the fine details.

In this case, since $U_1 < U_2$, we can see that,

$$\frac{dV(t)}{dt} > 0, \quad t_0 + i\omega_2 + x\omega_2 \leq t \leq t_0 + (i+1)\omega_2$$

and

$$\frac{dV(t)}{dt} < 0, \quad t_0 + i\omega_2 < t < t_0 + i\omega_2 + x\omega_2$$

where i is any integer.

Consequently,

$$t_{max} = t_0 + \alpha\omega_2 \qquad 0 \leq \alpha \leq \beta'_m - 1 \tag{16a}$$

and

$$t_{min} = t_0 + \beta\omega_2 + x\omega_2 \qquad 0 \leq \beta \leq \beta'_m - 1 \tag{16b}$$

where, α and β are integer variables.

Using Eqs. 11, 16a and 16b, and following an argument similar to that given in Appendix I, we can demonstrate the following proposition.

Proposition IIIb. The local maxima and the local minima of $I(t)$ for $r < 1$, are given by,

$$V^+(\alpha) = \frac{m}{2}(1 - x_1)V_2 + \frac{1}{2}(1 - x)V_2 + \sum_{n=1}^{\infty} \frac{U_2\omega_2}{n^2\pi^2}\left[mr \text{ Sin } n\pi x_1 \text{ Sin } 2n\pi\left(\alpha p' + \frac{y}{m} - \frac{x_1}{2}\right) - \text{Sin } n\pi x \text{ Sin } 2n\pi\left(y + \frac{x}{2}\right)\right]$$

and

$$V^-(\beta) = \frac{m}{2}(1 - x_1)V_2 - \frac{1}{2}(1 - x)V_2 + \sum_{n=1}^{\infty} \frac{U_2\omega_2}{n^2\pi^2}\left[mr \text{ Sin } n\pi x_1 \text{ Sin } 2n\pi\left(\beta p' + \frac{x+y}{m} - \frac{x_1}{2}\right) - \text{Sin } n\pi x \text{ Sin } 2n\pi\left(y + \frac{x}{2}\right)\right]$$

respectively, where $p' = 1/\beta'_m$.

One can prove Proposition IV for this case also by a proof similar to the one given in Appendix II, providing we recognize that $m = \beta'_m/\beta_m = p/p'$.

The development of the expressions for V_{min} and V_{max} follows the same logic as in the previous case. We have illustrated the derivation for V_{max} in Appendix III in detail. We merely state the results in the following proposition which can be used in conjunction with Proposition V.

Proposition VI.

The results of proposition V are valid for $r < 1$, if the definitions $z_1 = \min[ry', px]$ and $z_2 = \min[r(p - y'), px]$ are used.

A few properties of the functions, $V_{max}(y)$ and $V_{min}(y)$ are worth noting and are summarized in the following lemmas.

Lemma II.

$V_{max}(y)$ and $V_{min}(y)$ both are concave and piecewise linear functions with minimum values at $y = 0$ and $y = 1$ and maximum values at $y = 1 - x$.

Lemma III.

$V_{min}(y)$ has the maximum value of zero and also,

$$V_{min} = 0 \quad (1 - x) - p(1 - x) \le y \le (1 - x) + px,$$

$$V_{min} = 0, \quad (1 - x) - p(1 - x_1) \le y \le (1 - x); \quad r < 1$$

$$r \ge 1$$

Typical plots of $V_{max}(y)$ and $V_{min}(y)$ are shown in Figures 3a and 3b respectively.

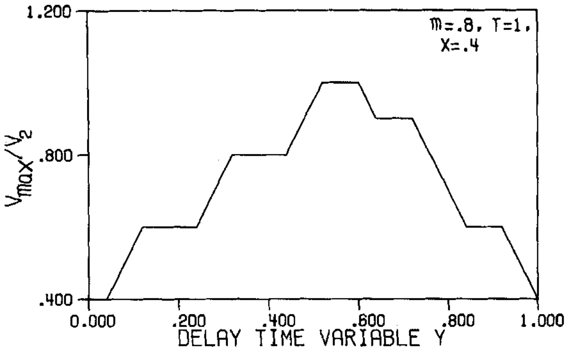

Figure 3a. V_{max} as a function of delay time.

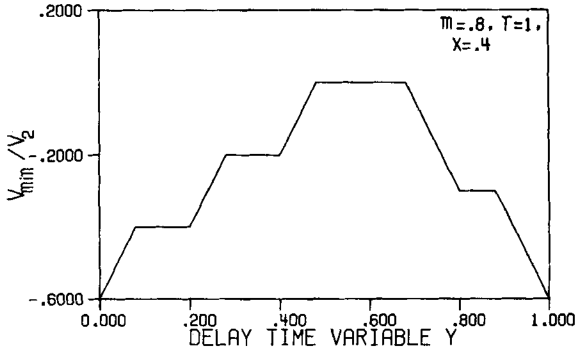

Figure 3b. V_{min} as a function of delay time.

Remarks

While these results were developed in terms of the homogeneous batch/semicontinuous configurations, they are applicable to all other equipment configurations. The only changes required are changes in the interpretation of key variables.

1. In semicontinuous-semicontinuous arrangement, the fractions x and x_1 are defined as $x = (T_S)_2/\omega_2$ and $x_1 = (T_S)_1/\omega_1$. U_1 and U_2 are the fixed processing rates of semicontinuous units. V_1 and V_2 are the amounts of material processed by Units 1 and 2 in their respective fixed processing times.

2. For an upstream semicontinuous unit and downstream batch unit, U_1 is the rate of processing of the semicontinuous unit while U_2 is the rate of filling of the batch unit. The fractions x and x_1 in this case are $x = (T_f)_2/\omega_2$ and $x_1 = (T_S)_1/\omega_1$.

If the upstream unit is a batch unit, U_1 is the rate of emptying the batch unit and x and x_1 are, $x = (T_S)_2/\omega_2$ and $x_1 = (T_e)_1/\omega_1$.

3. A continuous unit is but a special case of a semicontinuous unit in which x or x_1, depending upon the position (downstream or upstream) of the continuous unit in the arrangement, has a value of unity. In this case, ω for both units is equal to the cycle time of the noncontinuous equipment. Thus, $m = 1, p = 1$, and the volume of the vessel will be independent of y.

RESULTS AND DISCUSSION

By redefining the variables z_1 and z_2, we can condense the key results of this work to the formulae shown in Table 1. These results include two known special cases. In addition, we will summarize a few key properties of the storage volume function $V^*(y)$ and will show how these properties can be used to guide the selection of the optimal delay time. The connection between our results and the formulae derived by Takamatsu et al. (1982) for the special cases considered by the authors is outlined in Appendix IV.

Reduction to Special Cases

Consider the application of the results of Table 1 to the following two cases for which the optimal solutions are known. The first case is the degenerate one of $\omega_1 = \omega_2$, $U_1 = U_2$ and $t_0 = 0$, for which no storage vessel is required. The second case is that of an intermediate storage vessel linking one continuous unit with rate U_1 to one batch unit, which was considered by Takamatsu et al. (1979).

As one would expect, if the two units form a subtrain, the expressions in Table 1 do predict that the storage vessel can be of zero volume. It can be shown that $\beta_m = 1$, $\beta'_m = 1$; therefore, $p = 1/\beta_m = 1$. As $y = 0$, we have $k_2 = -1$ and $y' = 1 - x$. Consequently, $z_2 = x$ and substitution in the expression for $V_{max}(y \le 1 - x)$ leads to $V_{max} = 0$. Similarly it can be shown that $V_{min} = 0$. Thus, we confirm that no intermediate storage vessel is required.

Next, consider the case of a continuous unit followed by a batch unit. As noted earlier, for an upstream continuous unit, $x_1 = 1$ and $\omega_1 = \omega_2$. Clearly, we have $p = 1$, $k = 1$ and $\delta_1 = 0$. Since, $m = 1$, $r = x$ or $r < 1$. Note that in the definition of y for V_{max}, $k_2 = 0$ and $y = y'$. Thus, the definition of z_2 in Table 1 reduces to $z_2 = r(1 -$

TABLE 1. RESULTS FOR THE CALCULATION OF V_{max}, V_{min} AND V^*

Function	Definitions	Range	Expression
V_{max}	$y = k_2p + \delta_1 + y'$; $0 \le y' < p$; k_2 integer; $z_2 = \min[(p - y'), r(p - y'), px]$	$y \le 1 - x$	$m(1 - x_1)V_2 + U_2\omega_2[-z_2 + px + x(k_2p + \delta_1)]$
		$y \ge 1 - x$	$m(1 - x_1)V_2 + U_2\omega_2[-z_2 - y' + px + (1 - x)(1 - k_2p - \delta_1)]$
V_{min}	$x + y = k_1p + y'$; $0 \le y' < p$; k_1 integer; $z_1 = \min[y', ry', px]$	$y \le 1 - x$	$U_2\omega_2[z_1 - x(1 - k_1p)]$
		$y \ge 1 - x$	$U_2\omega_2[z_1 - y' + (1 - x)(1 - k_1p)]$

Volume of the storage vessel $V^* = V_{max} - V_{min}$.

y). Substituting into the expressions for V_{max} in Table 1, we have,

$$V_{max} = yV_2 \qquad \text{for } y \leq 1 - x$$
and
$$V_{max} = U_2\omega_2(1-r)(1-y) \qquad \text{for } y \geq 1 - x$$

For V_{min}, we have $k_1 = 0$, $y' = x + y$ and $z_1 = r(x + y)$ for $y \leq 1 - x$ while $k_1 = 1$, $y' = x + y - 1$ and $z_1 = r(x + y - 1)$ for $y \geq 1 - x$. Substituting these into the expressions for V_{min} in Table 1, we have,

$$V_{min} = yV_2 - (1-x)V_2 \qquad \textit{for } y \leq 1 - x$$
$$V_{min} = U_2\omega_2(1-r)(1-y) - (1-r)V_2 \qquad \textit{for } y \geq 1 - x$$

Combining the previous relations, we get,

$$V^* = V_{max} - V_{min} = (1-r)V_2 \qquad \text{for } 0 \leq y < 1.$$

Therefore, the size of the required storage vessel is $(1 - U_1/U_2)V_2$ which is the result derived by Takamatsu et al. (1979). As expected, the tank size required is independent of the delay time.

Properties of $V^*(y)$. We next state some important properties of $V^*(y)$, the difference between $V_{max}(y)$ and $V_{min}(y)$. Since, $V_{max}(y)$ and $V_{min}(y)$ are piecewise linear functions (Lemma II), the following lemma is an obvious consequence of Proposition IV.

Lemma IV. $V^*(y)$ is a piecewise linear and periodic function. From the definition of $V^*(y)$, it is clear that in general we have to superimpose the variations of V_{max} and V_{min} to evaluate $V^*(y)$. However, Lemma III is useful in the development of $V^*(y)$, because in the ranges given in Lemma III $V^*(y) = V_{max}(y)$ and $V_{min}(y)$ can itself be readily constructed. However, for the case $r < 1$, $V_{min}(y)$ should be included in calculating $V^*(y)$ for the range $(1-x) \leq y \leq (1-x) + px_1$, since $V_{min}(y) = -U_2\omega_2(1-r)(x + y - 1)$ in this range. If we include this additional portion in the ranges of Lemma III, we get,

$$(1-x) - p(1-x) \leq y \leq (1-x) + px \quad \text{for } r \geq 1 \quad (17a)$$

$$(1-x) - p(1-x_1) \leq y \leq (1-x) + px_1 \quad \text{for } r < 1 \quad (17b)$$

as the ranges of length p. This is useful, since once we construct $V^*(y)$ in these ranges, we also get the whole profile of $V^*(y)$ due to the periodicity of $V^*(y)$. Finally, because of the concave nature of $V_{max}(y)$ (Lemma II), the following important result can be stated.

Lemma V. $V^*(y)$ takes on its maximum value at $y = 1 - x \pm ip$, i integer, while it takes on its minimum value at $y = (1-x) - p(1-x) \pm ip$ for $r \geq 1$ and at $y = (1-x) - p(1-x_1) \pm ip$ for $r < 1$.

Because of the piecewise linear nature of V^*, it is possible to have other values of y which would also correspond to extreme values of V^*. However, the usefulness of the values of y given in Lemma V lies in the fact that we can readily calculate the extreme values of V^* at these points. The nature of $V_{max}(y)$ in the ranges of Eqs. 17a and 17b also reflects the following properties of $V^*(y)$.

Lemma VI. Except at the corner points, the slope of $V^*(y)$ takes on one of only the three values: 0 and $\pm$ $[\min(r,1)]U_2\omega_2$.

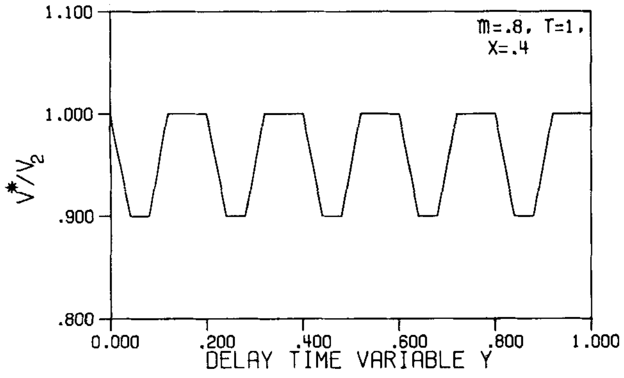

Figure 4. Tank size, V^*, as a function of delay time.

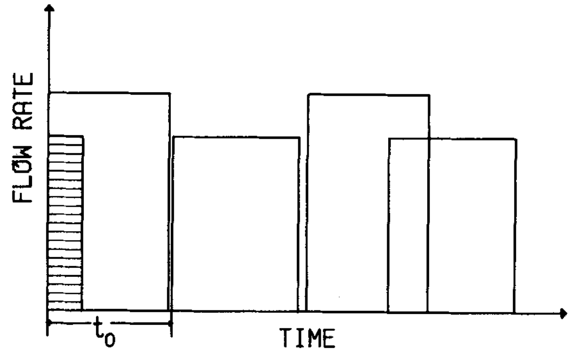

Figure 5. Input and output functions of the storage vessel for $y > 1 - x$.

Moreover, $V^*(y)$ either represents a corner point or has a zero slope when $V^*(y)$ is at its extreme values while it has a nonzero slope at its intermediate values.

A typical plot of $V^*(y)$ illustrating this structure is shown in Figure 4.

Choice of Delay Time

Given these observations of the behavior of $V^*(y)$, a suitable or an optimal value of the delay time can readily be selected. First of all, the ranges on y and r given in Lemma III over which $V_{min} = 0$ are appropriate to use when the operation is started with an empty storage vessel. With this choice of y the continuity of operation will not be disturbed due to overflowing or exhaustion of stored material. For all other choices of y, outside of the ranges given in Lemma III, the operation will need to be started with some initial inventory given by $V(0) = -V_{min}$. Therefore, for operation with no initial inventory the choice of $y = (1-x) - p(1-x)$ for $r \geq 1$ or $y = (1-x) - p(1-x_1)$ for $r < 1$ is optimal because it will require minimum V^*.

By studying the behavior of $V^*(y)$ in the ranges of Lemma III, one can find out other values of y in the same ranges which will also require minimum V^*. These other values will be in the neighbourhood of either of the ends of these ranges. Alternately, for a given size of storage vessel and its initial inventory, one can find out a suitable value of y so that the vessel will not overflow or run out of stored material. This can be done by choosing a y such that the following are satisfied.

$$V(0) + V_{max} \leq V^* \quad \text{and } V(0) + V_{min} \geq 0 \qquad (18a,b)$$

It is obvious that one can also estimate the maximum and the minimum hold up possible in a given tank, if the operation is started with a given value of y and $V(0)$.

One very important additional fact to be noted is that when a periodic operation is started, it is advisable to choose y such that $y \leq 1 - x$. This is because when $y > 1 - x$, the first cycle $(0 \leq t \leq \beta_m\omega_1)$ is not completely periodic because, as shown in Figure 5, the shaded portion of the operation is missing when the two units are started for the first time. The effect of this incompleteness will be that the holdup in the storage vessel will be increased uniformly by an amount $U_2\omega_2[y - (1-x)]$. If this amount can be somehow removed from the vessel in its first cycle, then the operation will experience no effect due to $y > 1 - x$. Because of the nature of $V_{max}(y)$ and $V_{min}(y)$ as given by Lemma II, there are two values of y which give the same values of V_{max} and V_{min}, i.e., one with $y \leq 1 - x$ and the other with $y \geq 1 - x$; since the choice of $y > 1 - x$ does not offer any particular advantage, it is advisable to have $y \leq 1 - x$.

Extension

Several other interesting cases can also be represented within the framework of the present analysis. For instance, there are processes in which apart from the main streams, i.e., those to and from the linked units, there are extraneous inputs and outputs to and from the storage vessel. The present analysis can be applied to this case, if all the inputs and outputs to and from the storage vessel occur simultaneously with their respective main streams and last for exactly the same durations as those of their respective main streams. Thus, one can write, $aV_1/\omega_1 = bV_2/\omega_2$ and $m = aV_1/bV_2 = \omega_1/\omega_2$, where a and b are the correction factors applied to the main streams to account for the extraneous inputs and outputs. Here, V_1 and V_2 as used in the results will be replaced by aV_1 and bV_2 and similarly U_1 and U_2 in the analysis will be replaced by aU_1 and bU_2 respectively.

If N_1 parallel, identical upstream units are emptied simultaneously and N_2 parallel, identical downstream units are filled simultaneously, then the present analysis can be applied with the modifications: $V_1 = N_1v_1$, $V_2 = N_2v_2$, $U_1 = N_1u_1$ and $U_2 = N_2u_2$.

For a stage with N parallel units emptied immediately one after another with the same rate and the whole stage operated in a periodic manner, if T_e is the time for emptying one unit, then the cycle time of the stage will be $N(T_e)$ or the cycle time of one unit, whichever is greater. If $N(T_e)$ is greater than the cycle time of one unit, the stage is equivalent to a continuous unit while in the other case it is equivalent to a batch unit with capacity Nv, cycle time ω, emptying time $N(T_e)$ and output flow rate u. A stage being filled can also be treated similarly.

In brief, if input and output patterns of an intermediate storage vessel can be structured to resemble the functional forms of $F_1(t)$ and $F_2(t)$ in a periodic operation, one can use our results for analysis of such a system. As an example, one can easily apply the present analysis in the case of a storage tank which receives input periodically from trucks of a fixed size and from which material is withdrawn periodically in batches of fixed size for processing.

EXAMPLE

An intermediate storage vessel acts as a surge vessel between one upstream batch unit and one downstream semicontinuous unit. Calculate the maximum and minimum values of the minimum storage capacity required. If the initial inventory in the storage vessel is 400 L and the storage vessel has a capacity of 6,800 L, choose a suitable timing of the operation to maintain continuous processing.

Upstream Batch Unit. $V_1 = 8{,}000$ L, $(T_f)_1 = 2.5$ h, $(T_B)_1 = 7.5$ h, $(T_e)_1 = 2.5$ h, $(T_p)_1 = 0.0$ h, $\omega_1 = 12.5$ h.

Downstream Semicontinuous Unit. $V_2 = 3{,}200$ L, $(T_S)_2 = 3.75$ h, $(T_i)_2 = 1.25$ h, $\omega_2 = 5$ h.

From the above data, $m = 2.5$, $x_1 = 0.2$, $x = 0.75$, $mx_1 = 0.5$ and $r = 3.75$. The smallest integer which when multiplied by m gives an integer is $\beta_m = 2$; hence, $p = 1/\beta_m = 0.5$, $px = 0.375$ and $p(1 - x) = 0.125$. Subtracting the integral multiples of p from mx_1, we get, $\delta_1 = 0.0$. Since, $r > 1$, we know from Lemma V that the minimum and the maximum of V^* occur at $y = (1 - x) - p(1 - x)$ and at $y = 1 - x$ respectively.

Minimum Tank Size and Maximum Tank Size

As $y = (1 - x) - p(1 - x) = 0.125$, $V^* = V_{\max}$, from Lemma III. For $V_{\max}$, y - $k_2p + \delta_1 + y'$ (Table 1); therefore, subtracting the integral multiples of p from $y - \delta_1 = 0.125$, we have $k_2 = 0$, $y' = 0.125$ and $p - y' = 0.375$. Since $r > 1$ and $p - y' = px$, $z_2 = px$ from Table 1. Substituting these values in the expression for $V_{\max}(y \leq 1 - x)$, we obtain minimum tank size = 6,400 L. For maximum tank size, we have $y = 1 - x = 0.25$, $k_2 = 0$, $y' = 0.25$ and hence $z_2 = p - y'$. After substitution, we obtain, maximum tank size = 6,934 L.

Choice of y for Continuity

As discussed earlier, we need to satisfy Eqs. 18a,b i.e.

$$V_{(0)} + V_{\max} \leq 6{,}800 \text{ and } V(0) + V_{\min} \geq 0$$

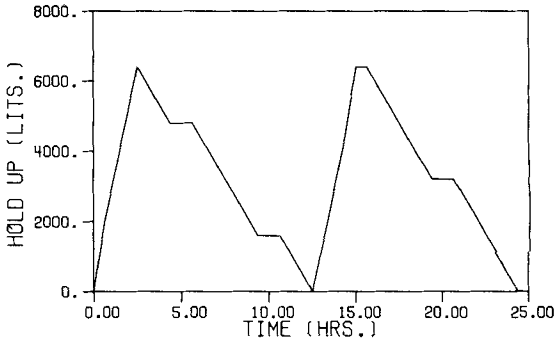

Figure 6. Variation of holdup in the storage vessel for y = 0.125.

therefore, $V_{\max} \leq 6{,}400$ and $V_{\min} \geq -400$. From previous calculations for minimum tank size and Lemma II, we can easily see that, $V_{\max} \leq 6{,}400$, if $y \leq 0.125$. Now, $V_{\min} = 0$ at $y = 0.125$. From Table 1, one can show that $V_{\min}$ will decrease at a rate of $-U_2\omega_2$ for $0.125 - p(1 - x) \leq y \leq 0.125$. Therefore, $V_{\min} = -400$ at $y = 0.09375$ and from Lemma II, it is clear that $V_{\min} \geq -400$, if $y \geq 0.09375$. Hence, the desired range of y is $0.09375 \leq y \leq 0.125$, i.e., if the delay time to t_0 is chosen in the range $0.469 \text{ h} \leq t_0 \leq 0.625$ h, continuity of operation can be maintained. The variation of the holdup volume for $y = 0.125$ is shown in Figure 6.

ACKNOWLEDGMENT

This work was supported by National Science Foundation Grant No. CPE79-24565.

NOTATION

a	= correction factor applied to V_1 to take into account the extraneous inputs to the storage vessel
b	= correction factor applied to V_2 to take into account the extraneous outputs from the storage vessel
$F_0(t)$	= function illustrated in Figure 2
$F_1(t)$	= input function to the storage vessel
$F_2(t)$	= output function from the storage vessel
i	= an integer
$I(t)$	= integral defined by Eq. 8
j	= an integer
k	= an integer
k_1	= integer used in the definition of y' for $V_{\min}$ (Table 1)
k_2	= integer used in the definition of y' for $V_{\max}$ (Table 1)
m	= capacity ratio defined by Eq. 1
n	= integer variable used in series summation
N	= number of parallel units in a stage
p	= period of $V^*(y)$ defined as $p = 1/\beta_m$
p'	= defined as $1/\beta'_m$
r	= ratio of input rate and output rate, to and from the storage vessel, Eqs. 3a and 3b
t	= time
$t_{\max}$	= time representing a local maximum in $I(t)$
$t_{\min}$	= time representing a local minimum in $I(t)$
t_0	= the difference between the moment when Unit 2 starts getting filled for the first time from the storage vessel and the moment when the storage vessel starts getting filled for the first time from Unit 1
T_e	= time required to empty a batch unit
T_f	= time required to fill a batch unit
T_i	= shutdown time for a semicontinuous unit
T_p	= preparation time and waiting time for a batch unit
T_B	= processing time of a batch unit
T_S	= processing time for a semicontinuous unit
u	= rate of input or output, to or from an individual unit within a stage of N parallel identical units

U	= rate of input or output, to or from, the storage vessel
U_0	= value of the function $F_0(t)$ in Figure 2
v	= the capacity of an individual unit within a stage of N parallel, identical units
$V(t)$	= holdup in the storage vessel
V_{max}	= maximum value of $I(t)$
V_{min}	= minimum value of $I(t)$
V_0	= area under the curve for one cycle of $F_0(t)$ in Figure 2
V_1	= batch size for the upstream unit
V_2	= batch size for the downstream unit
V^*	= minimum size of the storage vessel required to decouple the upstream and the downstream units
$V^+(\alpha)$	= represents a local maximum in $I(t)$
$V^-(\beta)$	= represents a local minimum in $I(t)$
x	= filling fraction for the downstream batch unit while processing fraction for the downstream semicontinuous unit
x_0	= fraction defined for $F_0(t)$ in Figure 2
x_1	= emptying fraction for the upstream batch unit while processing fraction for the upstream semicontinuous unit
y	= fractional delay time variable, t_0/ω_2
y'	= a variable defined for the calculation of V_{max} and V_{min} in Table 1
z	= a variable
z_1	= a variable defined for the calculation of V_{min}
z_2	= a variable defined for the calculation of V_{max}

Greek Letters

α	= an integer variable used for V_{max}
β	= an integer variable used for V_{min}
β_m	= minimum integer which when multiplied by m gives an integer value
β'_m	= defined by equation $\beta_m\omega_1 = \beta'_m\omega_2$
β'	= a dummy integer variable
ω	= cycletime of a unit
ω_0	= cycletime of $F_0(t)$ in Figure 2
δ_1	= defined by equation $mx_1 = kp + \delta_1$ such that $0 \le \delta_1 < p$ for an integer k
τ	= a dummy variable

Subscripts

1	= refers to the upstream unit
2	= refers to the downstream unit

Mathematical Symbols

min { }	= minimum of the quantities within the bracket
| |	= absolute value

APPENDIX I: PROOF OF PROPOSITION IIIa

Substituting $t = t_{max}$ from Eq. 14a in Eq. 13 and using Eq. 12, we get,

$$V^+(\alpha) = m(1-x_1)V_2 - \sum_{n=1}^{\infty} \frac{U_2\omega_2}{n^2\pi^2} \operatorname{Sin} n\pi x \times \left[\operatorname{Sin} 2n\pi\left(m\alpha + mx_1 - y - \frac{x}{2}\right) + \operatorname{Sin} 2n\pi\left(y + \frac{x}{2}\right)\right]$$

Now, mx_1 can be written as, $mx_1 = kp + \delta_1$ such that $0 \le \delta_1 < p$ for an integer k. Since $m = \beta'_m/\beta_m$, $m\alpha = \beta'_m\alpha p$. From preceding relations, we get,

$$V^+(\alpha) = m(1-x_1)V_2 - \sum_{n=1}^{\infty} \frac{U_2\omega_2}{n^2\pi^2} \operatorname{Sin} n\pi x \times \left[\operatorname{Sin} 2n\pi\left[(\beta'_m\alpha + k)p + \delta_1 - y - \frac{x}{2}\right] + \operatorname{Sin} 2n\pi\left(y + \frac{x}{2}\right)\right]$$

We have restricted α such that $0 \le \alpha \le \beta_m - 1$. As α takes β_m unique values, $(\beta'_m\alpha + k)$ also takes β_m unique values, therefore $(\beta'_m\alpha + k)$ can always be represented as, $\beta'_m\alpha + k = i\beta_m + \alpha'$ such that $0 \le \alpha' \le \beta_m - 1$ for an integer i. Since the Sine function has a period of 2π, substituting the previous relation in $V^+(\alpha)$ and denoting the dummy integer variable α' again as α, we obtain the result for $V^+(\alpha)$ in proposition IIIa. Result for $V^-(\beta)$ can be derived by following a similar procedure.

APPENDIX II: PROOF OF PROPOSITION IV

Define $V^{**}(\alpha,\beta) = V^+(\alpha) - V^-(\beta) \qquad 0 \le \alpha,\beta \le \beta_m - 1$

There are β_m values of α and β each, so $V^{**}(\alpha,\beta)$ can take on β_m^2 values. From Eqs. 13 and 15a,b,

$$V^* = \max_{\alpha} V^+(\alpha) - \min_{\beta} V^-(\beta) = \max_{\alpha,\beta} V^{**}(\alpha,\beta) \qquad \text{(A1)}$$

From Proposition IIIa,

$$V^{**}_{(\alpha,\beta)} = m(1-x_1)V_2 - \sum_{n=1}^{\infty} \frac{U_2\omega_2}{n^2\pi^2} \operatorname{Sin} n\pi x \times \left[\operatorname{Sin}\left(\alpha p + \delta_1 - y - \frac{x}{2}\right) - \operatorname{Sin} 2n\pi\left(\beta p - y - \frac{x}{2}\right)\right]$$

Let us consider two values of y, $y = y_1$, and $y = y_2 = y_1 + p$. Denote the corresponding values of $V^{**}(\alpha,\beta)$ by subscripts 1 and 2 respectively. But for arbitrary integers $\alpha = i$ and $\beta = j$, $0 \le i,j \le \beta_m - 1$, we have $V_1^{**}(i,j) = V_2^{**}(i',j')$ where, $i' \equiv i + 1$ and $j' \equiv j + 1$. Here, the symbol $\equiv$ indicates that $i + 1$ and $j + 1$ be reduced to the range $0 \le i',j' \le \beta_m - 1$ by subtracting an integral multiple of β_m, if required.

Since, our choice of i and j was arbitrary, we can conclude that for any given integers i,j; $0 \le i,j \le \beta_m - 1$; for $V_1^{**}(\alpha,\beta)$, we can always choose integers $i' \equiv i + 1$, $j' \equiv j + 1$; $0 \le i',j' \le \beta_m - 1$; such that $V_1^{**}(i,j) = V_2^{**}(i',j')$. Therefore, set of values of $V_1^{**}(\alpha,\beta)$ and $V_2^{**}(\alpha,\beta)$ are same. Consequently, their maximum values are same and hence from Eq. A1, $V^*|_{y=y_1} = V^*|_{y=y_1+p}$. Our choice of $y = y_1$ was also arbitrary, so $V^*(y)$ is a periodic function with period p.

APPENDIX III: PROOF OF PROPOSITION VIb

From Proposition IIIb, expressing $V^+(\alpha)$ in Cosine form,

$$V^+(\alpha) = \frac{m}{2}(1-x_1)V_2 + \frac{1}{2}(1-x_1)V_2 + \sum_{n=1}^{\infty} \frac{U_2\omega_2}{2n^2\pi^2}\left[mr \operatorname{Cos} 2n\pi\left(\alpha p' + \frac{y}{m} - x_1\right) - mr \operatorname{Cos} 2n\pi\left(\alpha p' + \frac{y}{m}\right) - \operatorname{Cos} 2n\pi y + \operatorname{Cos} 2n\pi(x+y)\right]$$

Adding and subtracting x_1 from the second cosine term,

$$V^+(\alpha) = \frac{m}{2}(1-x_1)V_2 + \frac{1}{2}(1-x_1)V_2 + \sum_{n=1}^{\infty} \frac{U_2\omega_2}{2n^2\pi^2}\left[mr \operatorname{Cos} 2n\pi\left(\alpha p' + \frac{y}{m} - x_1\right) - mr \operatorname{Cos} 2n\pi\left(\alpha p' + \frac{y}{m} - x_1 + x_1\right) - \operatorname{Cos} 2n\pi y + \operatorname{Cos} 2n\pi(x+y)\right]$$

Now, $mx_1 = kp + \delta_1$ and $m = \beta'_m/\beta_m = p/p'$; hence, $x_1 = kp' + \delta_1/m$. Substituting for x_1 and merging kp' into $\alpha p'$ as done in previous appendix, we get,

$$V^+(\alpha) = \frac{m}{2}(1-x_1)V_2 + \frac{1}{2}(1-x_1)V_2 + \sum_{n=1}^{\infty}\frac{U_2\omega_2}{2n^2\pi^2}\left[mr \operatorname{Cos} 2n\pi\left(\alpha p' + \frac{y-\delta_1}{m}\right) - mr \operatorname{Cos} 2n\pi\left(\alpha p' + \frac{y-\delta_1}{m} + x_1\right) - \operatorname{Cos} 2n\pi y + \operatorname{Cos} 2n\pi(x+y)\right]$$

Applying the transformation $y = k_2p + \delta_1 + y'$ and merging k_2p with αp, we obtain,

$$V^+(\alpha) = \frac{m}{2}(1-x_1)V_2 + \frac{1}{2}(1-x_1)V_2 + \sum_{n=1}^{\infty}\frac{U_2\omega_2}{2n^2\pi^2}\left[mr \operatorname{Cos} 2n\pi\left(\alpha p' + \frac{y'}{m}\right) - mr \operatorname{Cos} 2n\pi\left[\alpha p' + \frac{y'}{m} - (1-x_1)\right] - \operatorname{Cos} 2n\pi y + \operatorname{Cos} 2n\pi(x+y)\right]$$

Since, $0 \le y'/m < p'$ and $0 \le \alpha \le \beta_m' - 1$, $\alpha p' + y'/m < 1$ so it is clear that Eq. 9 can be applied using which for $y \le 1 - x$, gives

$$V^+(\alpha) = m(1-x_1)V_2 + \frac{U_2\omega_2}{2}\left[mr\left|\alpha p' + \frac{y'}{m} - (1-x_1)\right| + mr\left[\alpha p' + \frac{y'}{m} - (1-x_1)\right] - 2x\alpha p + 2x(k_2p + \delta_1)\right]$$

Dividing α values into two parts, $\alpha p' + y'/m - (1 - x_1) \ge 0$ and $\alpha p' + y'/m - (1 - x_1) < 0$, and proceeding in exactly the same manner as in the text, we get the following expressions.

1. $y \le 1 - x$

$$V_{\max} = m(1-x_1)V_2 + U_2\omega_2[x(k_2p + \delta_1)], \quad y' \le p(1-x_1)$$
$$V_{\max} = m(1-x_1)V_2 + U_2\omega_2[r[y' - p(1-x_1)] + x(k_2p+\delta_1)], \quad y' \ge p(1-x_1)$$

2. $y \ge 1 - x$

$$V_{max} = m(1-x_1)V_2 + U_2\omega_2[r[y' - p(1-x_1)] - y' + (1-x)(1-k_2p-\delta_1)], \quad y' \ge p(1-x_1)$$
$$V_{\max} = m(1-x_1)V_2 + U_2\omega_2[-y' + (1-x)(1-k_2p-\delta_1)], \quad y' \le p(1-x_1)$$

Defining $z_2 = \min[r(p - y'), px]$, we get the result in Proposition VIb.

APPENDIX IV: REDUCTION TO RESULTS OF TAKAMATSU et al. (1982)

Theorem 2 of Takamatsu et al. (1982), states that for the case $U_1 \ge U_2$, $V(0) = 0$, and $V_{\min} = 0$, the delay time of $(1 - P/U_2)(M - 1)GCM\ (\omega_1,\omega_2)$ yields the minimum tank size for the batch-batch configuration. Moreover, for $U_1 = U_2$, that tank size is given by

$$V^* = \{\text{trunc}\ (Q) + \min\ [\text{mod}\ (Q, 1 \cdot U/P, 1]\} \cdot GCM(V_1,V_2) \quad \text{(A2)}$$

where $Q = (1 - P/U)(M + N - 2)$, $N = V_1/GCM(V_1,V_2)$

$M = V_2/GCM(V_1,V_2)$ and P is the rate of production.

In our notation, $M = \beta m$, $N = \beta_m'$ and $P - U_2x = U_1x_1$. Lemmas III and IV immediately yield the aforementioned value of the delay time, namely,

$$y = (1-x)(1-p)$$

For the case $r = 1$ and $V_{\min} = 0$, Table 1 prescribes the tank size

$$V^* = [m(1-x) + y - y']^{1/2} \quad \text{for } y' \le p(1-x) \quad \text{(A3)}$$

and

$$V^* = [m(1-x) + y + (1-x)(y'\ p)/x]V_2 \quad \text{for } y' > p(1-x) \quad \text{(A4)}$$

Using $y = (1 - x)(1 - p) = k_2p + \delta_1 + y'$ and $mx = kp + \delta_1$, it can readily be shown that

$$Q = (k_2 - k + \beta' - 1) + (\beta_m y' + x)$$

In addition, for $y' < p(1 - x)$ it follows that

$$\beta_m y' + x < 1$$
$$\text{Trunc}(Q) = k_2 - k + \beta_m' - 1$$

and

$$\text{mod}\ (Q,1) - \beta_m y' + x$$

while for $y' \ge p(1 - x)$, it similarly follows that

$$\beta_m y' + x \ge 1$$
$$\text{Trunc}(Q) = k_2 - k + \beta_m'$$

and

$$\text{mod}(Q, 1) = \beta_m y' + x - 1$$

Substituting these results into eq. A2, one readily obtains Eqs. A3 and A4. Thus, the results of Takamatsu et al. (1982), obtained using a different approach, are consistent with the results of Table 1.

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Manuscript received September 21, 1981; revision received August 2, and accepted August 30, 1982.